Large N limit of the Yang-Millls measure on closed surfaces

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BIRS workshop

Stochastics and Geometry

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Probabilities on Character varieties in high dimension

Γ	fixed group		
G_N	classical compact Lie groups	$\mathrm{rank}(G_N)$	$\to \infty$
	finite groups with	$\#G_N$	

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Qts: Assume G_N realised as $N \times N$ matrices

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$$\tau_N(\gamma) := \frac{1}{N}\mathrm{Tr}(X_N(\gamma)) \qquad \forall \gamma \in \Gamma$$

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Singer 97'
2D-Yang-Mills

Probabilities on Character varieties in high dimension

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Probabilities on Character varieties in high dimension

Γ $\mathbb{F}_r$ $\pi_1(\Sigma_g)$

Σ_g closed or. surface genus g

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Probabilities on Character varieties in high dimension

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G_N $\quad U(N), SU(N), O(N), USp(N)$ $\quad S_N$

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$$= \frac{\#Fix(X_N(\gamma))}{N}$$

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Ex. Prob. meas. on $\mathcal{X}_N$:

- Assume Γ finitely generated, G_N finite

then $\mathrm{Hom}(\Gamma, G_N)$ finite, $\quad X_N$ unif. RV on $\mathrm{Hom}(\Gamma, G_N)$

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$\Gamma = \pi_1(\Sigma_g)$ $G_N = S_N$ $\tilde{\Sigma}_g \times \{1, \ldots N\}/\Gamma$

Random N-sheeted covering of Σ_g

Probabilities on Character varieties in high dimension

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- $\Gamma = \mathbb{F}_r = \langle \ell_1, \ldots, \ell_r \rangle$

Sample $U_1, \ldots, U_r$ independent Haar distr. RV on G_N.

Set $X_N(\ell_i) := U_i$ $\forall i$

Extend multipli., $X_N : \mathbb{F}_r \to G_N$

e.g. $X_N(\ell_2 \ell_3^{-1} \ell_1^2) = X_N(\ell_2) X_N(\ell_3)^{-1} X_N(\ell_1)^2$

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$\Gamma \qquad \mathbb{F}_r \qquad \pi_1(\mathbb{G}) \qquad \pi_1(\Sigma_g)$

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$$\mathcal{X}_N = \mathrm{Hom}(\Gamma, G_N)/G_N$$

Ex. Prob. meas. on $\mathcal{X}_N$:

- $\Gamma = \pi_1(\Sigma_g), g \geq 2 \qquad G_N = SU(N)$

$\mathcal{X}_N^o = \mathrm{Hom}^{irrep}(\Gamma, G_N)/G_N$

finite volume symplectic manifold

X_N with law $\frac{1}{Z_g}\mathrm{vol}_{\mathcal{X}_N^o}$

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Random flat connection on $\tilde{\Sigma}_g \times \mathbb{C}^N/\Gamma$

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Probabilities on Character varieties in high dimension

$$\Gamma \qquad \mathbb{F}_r \longrightarrow \pi_1(\mathbb{G}) \longrightarrow \pi_1(\Sigma_g) \longrightarrow 1$$

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with r faces

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$\simeq \mathbb{F}_{2g+r-1}$
freely gen by $\ell_1, \ldots, \ell_{r-1}, x_1, \ldots, y_g$

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$(p_t)_{t>0}$ semi-group/heat kernel on G_N $\qquad p_t \rightharpoonup \delta_1$, $t \to 0$

$a \in {\mathbb{R}_+^*}^r \qquad p_t \circ Ad(g) = p_t$

Probabilities on Character varieties in high dimension

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Lemma: $\quad Z_{\mathbb{G},a} = Z_{g,T} \qquad$ where $T = \sum_f a_f$.

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Probabilities on Character varieties in high dimension

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$\mathbb{G}$ Graph embedded in Σ_g closed or. surface genus g

with r faces

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$$\mathcal{X}_N = \mathrm{Hom}(\Gamma, G_N)/G_N$$

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Lemma: $\quad Z_{\mathbb{G},a} = Z_{g,T} \qquad$ where $T = \sum_f a_f$.

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Probabilities on Character varieties in high dimension

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Pb: Concentration of μ_{ABG} as $N \to \infty$?

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Thm [Magee,Naud, Puder 2022]:

Random cover compact hyperbolic surface has w.h.p.

relative spectral gap $\frac{3}{16} - \varepsilon$ (conjecture $\frac{1}{4}$).

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$(p_T)_{T>0}$ heat kernel for metric

	$O(N)$	$U(N)$	$USp(N)$
β	1	2	4

$$\langle X, Y \rangle_N = \frac{\beta N}{2} \mathrm{Tr}(X^* Y) \quad \forall X, Y \in \mathfrak{g}_N$$

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Thm [Biane 97', Xu 97', Lévy, Sengupta & Anshelevich 11']:

$\forall \gamma \in \pi_1(\mathbb{G}), a \in {\mathbb{R}_+^*}^r$

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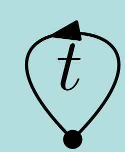
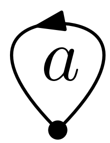
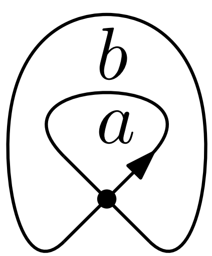
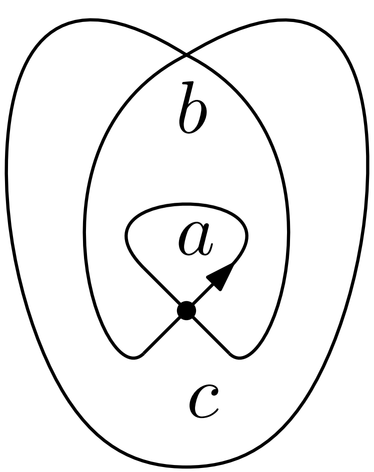
with $\tau_{\mathbb{G},a}$ deterministic, sol of $(KKMM)$, $\tau_{\mathbb{G},a}($  $) = e^{-\frac{t}{2}}$

Master field on discs $\mathbb{R}^2$ or $\mathbb{D}$

Kasakov-Kostov 81'

ℓ	$\tau(\ell)$
	$e^{-\frac{a}{2}}$
	$(1-a)e^{-a-\frac{b}{2}}$
	$e^{-\frac{3}{2}a-b-\frac{c}{2}}(1-3a+\frac{3}{2}a^2-b(1-a))$

NB: For $\ell \in \mathrm{L}(\mathbb{R}^2)$, $\tau(\ell)$ independent of $D \supset \ell$.

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Kasakov-Kostov 81' Anschelevich-Sengupta '11, Lévy 11'

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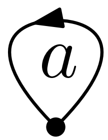
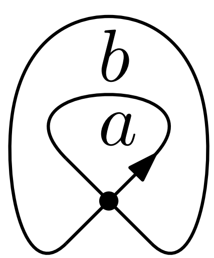
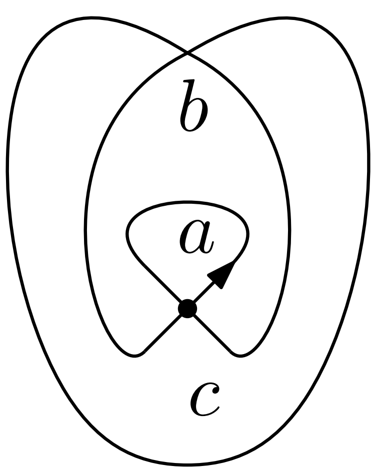
	$e^{-\frac{a}{2}}$
	$\tau(\ell^2) = (1-a)e^{-a}$
	$\tau(\ell^3) = (1 - 3a + \frac{3}{2}a^2)e^{-\frac{3}{2}a}$
	$\tau(\ell^n) = e^{-\frac{an}{2}} \frac{1}{n} \sum_{k=0}^{n-1} \frac{(-na)^k}{k!} \binom{n}{k+1}$
	$= e^{-\frac{na}{2}} \frac{1}{n} L_{n-1}^{(1)}(na)$
	$(1-a)e^{-a-\frac{b}{2}}$
	$e^{-\frac{3}{2}a-b-\frac{c}{2}}(1 - 3a + \frac{3}{2}a^2 - b(1-a))$

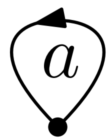
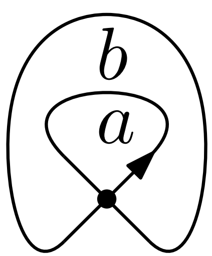
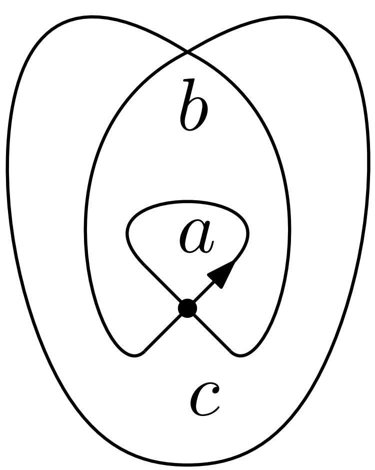
index 1 Laguerre polynomial

NB: For $\ell \in \mathrm{L}(\mathbb{R}^2)$, $\tau(\ell)$ independent of $D \supset \ell$.

Master field on discs $\mathbb{R}^2$ or $\mathbb{D}$

Kasakov-Kostov 81' Anschelevich-Sengupta '11, Lévy 11'

Hall 17', D.& Norris 17'

ℓ	$\tau(\ell)$
	$e^{-\frac{a}{2}}$
	$(1-a)e^{-a-\frac{b}{2}}$
	$e^{-\frac{3}{2}a-b-\frac{c}{2}}(1-3a+\frac{3}{2}a^2-b(1-a))$

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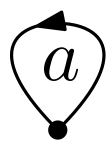

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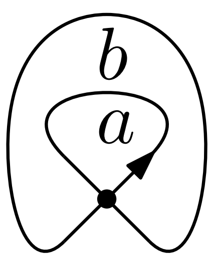

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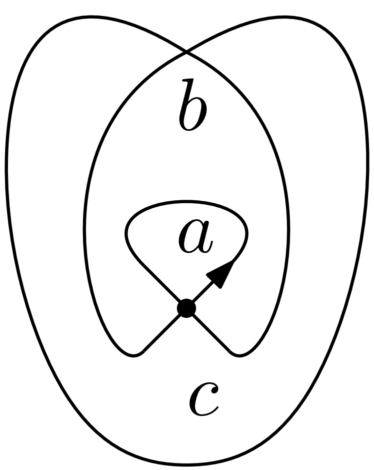

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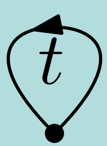
$\tau_{\mathbb{S}^2}($ 

$) = J_1(2\sigma) = \int_{-2}^{2} e^{i\sigma x} \frac{\sqrt{4-x^2}dx}{2\pi}$

$$\sigma = \sqrt{\frac{t(T-t)}{T}} \leq \frac{\pi}{2}$$

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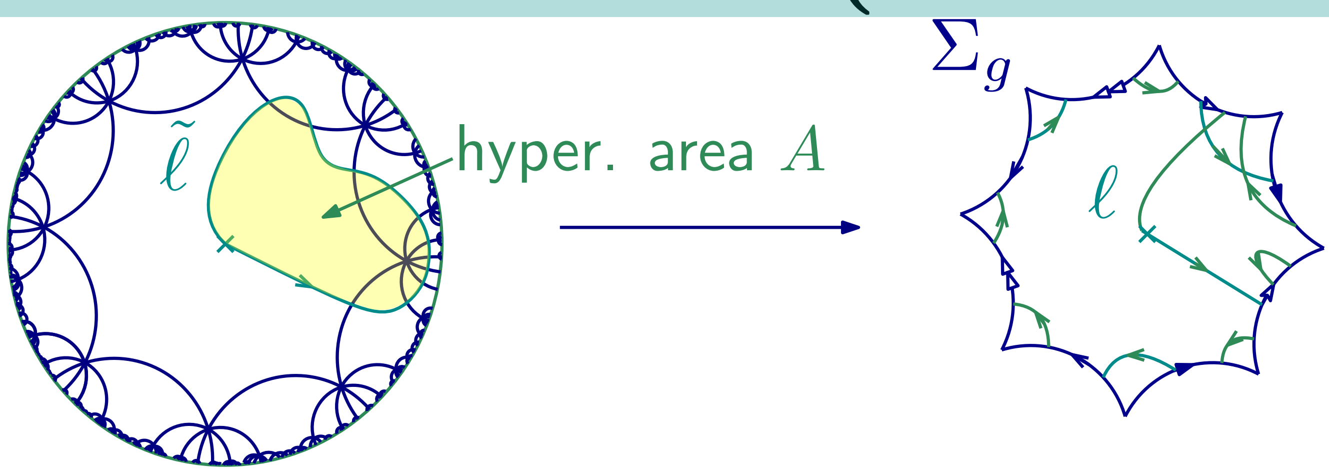

$\mathrm{tr}(H_\ell) \xrightarrow{\mathbb{P}} e^{-\frac{A}{2}}$

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Lemma: τ_{Σ_g} defines a positive funct. on $\mathrm{RL}_p(\Sigma_g)$:

$$\sum_{i,j} \alpha_i \overline{\alpha}_j \tau_{\Sigma_g}(\ell_i \ell_j^{-1}) \geq 0 \qquad \forall \ell_i \in \mathrm{RL}_p(\Sigma_g), \alpha_i \in \mathbb{C}$$

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- $\gamma = \alpha^n$ with α simple, $n \in \mathbb{Z}$
- $\gamma \subset \Sigma_1$ where $\Sigma = \Sigma_1 \# \Sigma_2$ with $\Sigma_2 \not\simeq \overline{\mathbb{D}}$

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Thm [D-Lemoine 22']: Conj. holds true iff it holds for all

$\gamma = \alpha^n \beta$, where

- α simple contractible, $n \in \mathbb{Z}$,
- β geodesic intersecting α only at its base point.

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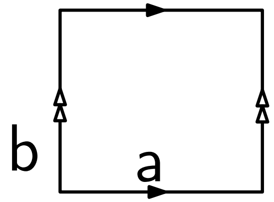

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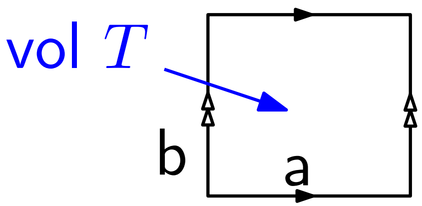

$$\Gamma = \mathrm{RL}(\mathbb{T}_2),$$

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	$O(N)$	$U(N)$	$USp(N)$
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$(p_T)_{T>0}$ heat kernel for metric

$$\langle X, Y\rangle_N = \frac{\beta N}{2}\mathrm{Tr}(X^* Y) \quad \forall X, Y \in \mathfrak{g}_N$$

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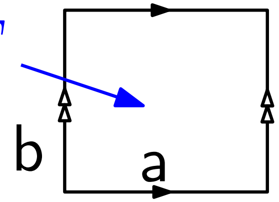

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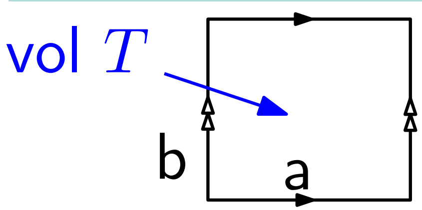

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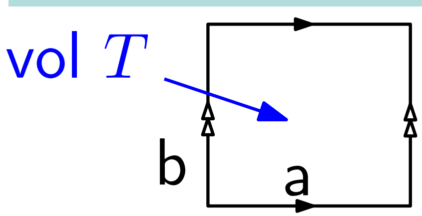

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$n_x = \#x - \#x^{-1}$ in w

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If $\Sigma = \Sigma_1 \# \Sigma_2$ along simple separating loop ℓ

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Idea:

Two homotopic loops can be deformed into one another by Makeenko-Migdal deformation iff

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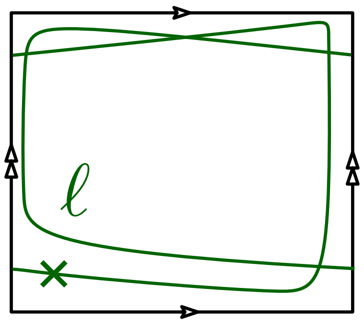

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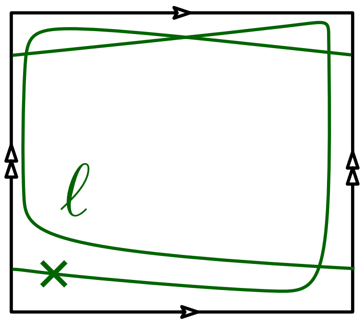

Idea: OK if done locally with a winding loop.

Argument 1+: Low weight representations

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When $G_N = SU(N)$,

$$\hat{G}_N \simeq \{\lambda \in \mathbb{Z}^N : \lambda_1 \leq \ldots \leq \lambda_N\}/(1, \ldots 1).\mathbb{Z}$$

$\hat{G}_N^{(k)}$ k-almost trivial irrep

Say λ is k-almost trivial if it has a represent with

$$-k \leq \lambda_1, 0 = \lambda_{k+1} = \ldots = \lambda_{N-k}, \lambda_N \leq k$$

Set $\quad p_T^{(k)}(g) = \sum_{\lambda \in \hat{G}_N^{(k)}} \dim(\lambda)\chi_\lambda(g)e^{-\frac{T}{2}C_\lambda},$

where χ_λ character λ

For $\mathbb{G}$ emb. graph, r faces, $a \in \mathbb{R}_+^r$

$$\mathrm{YM}_{\mathbb{G},a}^{(k)}(dh) = \prod_{f=1}^r p_{a_i}^{(k)}(h(\ell_i))\mu_{\mathbb{G}}(dh)$$

Lemma: For $g \geq 2$, $\mathbb{G}$ one face emb. in Σ_g, $T > 0$,

$\forall \alpha, \beta \in \pi_1(\mathbb{G})$, $\forall p \geq 1$, $\exists k \geq 1/$

$$\mathbb{E}_{\mathrm{YM}}[\tau_N(\alpha)\tau_N(\beta)] = Z_{g,T,SU(N)}^{-1} \int_{\mathcal{X}_N} \tau_N(\alpha)\tau_N(\beta)d\mathrm{YM}^{(k)} + O(N^{-p}).$$

Some questions

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  Chatterjee, Jafarov, Cao, Park, Pfeffer, Sheffield, Lemoine, Novak

"2D-QCD is a string theory"

Thank you!